

# Spin 1/2 and Invariant Coefficients

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## Abstract

In the quantum theory of fields one writes the relativistic field operator as a linear combination of annihilation operators, with invariant coefficient functions. The annihilation operators transform as physical, massive, single particle states with a unitary representation of the Poincaré group, while the relativistic field operator transforms with a nonunitary spin 1/2 representation of the homogeneous Lorentz group. The Lorentz group represents translations trivially, i.e. as multiplication by unity. Here the nonunitary representation is provided with translation matrices, so that the unitary and the nonunitary representations represent the same group, the Poincaré group. Translation matrix invariance is shown to give the free particle Dirac equation, without invoking parity. The coefficient functions for a given momentum determine a current. These currents turn out to be, within a constant factor, the electromagnetic vector potential of the free particle source moving with that momentum. Thus it is shown that the Dirac and Maxwell equations can be related to the inclusion of translation matrices in the transformations of field operators.

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# 1 Introduction

Successive infinitesimal rotations, boosts, and translations transform spacetime yet preserve the spacetime metric. There are many linear representations of these transformations and each representation has its own set of vectors that transform via that representation. This paper determines some of the mathematical consequences of requiring a vector of one type to be a linear combination of vectors of a second type with invariant coefficients. The two types of vectors transform via seemingly different representations, yet they can be joined in this way.

The vector to be constructed,  $\psi_l(x^\mu)$ , transforms like a relativistic field operator. It has a discrete index  $l$  that labels components that transform via a non-unitary finite dimensional matrix representation. The representation is the four component spin  $1/2$  representation conventionally designated as spin  $(0, 1/2) \oplus (1/2, 0)$ . There are also continuous parameters  $x^\mu$ ,  $\mu \in \{1, 2, 3, 4\}$ , labeling the field of values that transforms via a differential representation. The specific properties of relativistic field operators are not needed here; just the transformation rules are needed. Vector fields like the relativistic field operator that transform in this way are herein called ‘covariant field vectors.’

The vector  $\psi_l(x^\mu)$  is to be written as a linear combination of vectors  $a(p^\mu, \sigma)$  that transform differently. These second type vectors transform like annihilation operators. Components are labeled by a discrete index  $\sigma$  and take values as a field over the continuous parameters  $p^\mu$ . The representation is unitary with finite dimensional matrices, which are functions of the parameters  $p^\mu$ . The only properties needed here are the transformation properties of these vector fields, herein called ‘canonical field vectors.’

The continuous parameters of the covariant field are called the coordinates  $x^\mu$  while the parameters  $p^\mu$  of the canonical field are called momentum components. The field defined over coordinate space  $x^\mu$  transforms via a covariant (nonunitary) dimensional representation, while the field defined over momentum space  $p^\mu$  transforms via a canonical (unitary) transformation. Thus coordinates  $x^\mu$  and momenta  $p^\mu$  are distinguished by the transformation properties of their respective fields.

This paper derives some mathematical consequences of requiring a covariant vector  $\psi_l(x^\mu)$  to be a linear combination of canonical vectors  $a(p^\mu, \sigma)$  with invariant coefficients  $u_l(x^\mu, p^\nu, \sigma)$ . Abbreviated, this is  $\psi = \sum_a u a$ . When the covariant and canonical vectors transform by their respective covariant and canonical representations indicated by primes, the coefficients remain unchanged,  $\psi' = \sum_{a'} u a'$ . The coefficients are sometimes called intertwiners and can be understood as a kind of rectangular matrix transforming a basis in  $\{p^\mu, \sigma\}$  to a basis in  $\{x^\mu, l\}$ .

Obtaining the consequences of having invariant coefficient functions has been developed and investigated by many authors, [1],[2],[3]. Various covariant representations can be used.

The covariant representation here has translation matrices as well as the differential translation representation; others frequently have just the differential translation representation.

Some investigations assume the coefficient functions obey wave equations and use the constraints on coefficient functions to constrain and analyze wave equations,[2],[3]. Here no such assumption is made; neither Dirac's equation nor Maxwell's equations nor any other wave equation are explicitly imposed. Yet among the fields found here are fields that obey Dirac's equations and Maxwell's equations.

Obtaining solutions to Dirac's equation without assuming Dirac's equation has been done before, [4]. In that derivation, simplicity under spatial inversion is needed. Here parity and inversions are not considered, only transformations built from infinitesimal transformations and hence connected to the identity are discussed. The needed assumption here is based on there being two covariant representations differing by the choice of momentum matrices.

In this paper  $4 \times 4$  momentum matrices accompany the nonunitary  $4 \times 4$  matrices that represent spin  $1/2$  homogeneous Lorentz transformations.[5] The '12' and '21' notation is based on the fact that the momentum matrices can be written as triangular matrices. One representation has just the 12 block nonzero and is called the '12-representation' and the other has momentum matrices with just the 21 block nonzero, the '21-representation.' The other off-diagonal block and the diagonal blocks, i.e. the 11- and 22-blocks, are null for these momentum matrices.

The relationship between the 12-momentum matrices and the 21-momentum matrices induces a relationship between the 12- and 21-representations and hence a relationship between 12- and 21-covariant field vectors. The induced relationship is just a change of parity. The 12-covariant vector  $\psi_l^{(12)}(x^\mu)$  is associated with a 21-covariant vector  $\tilde{\psi}_l^{(21)}(x^\mu)$ . When one combines the translation-matrix-independent part of the 12-covariant field vector  $\psi_l^{(12)}(x^\mu)$  with the translation-matrix-independent part of the related 21 field  $\tilde{\psi}_l^{(21)}(x^\mu)$  one finds that the combination obeys the Dirac equation. Thus the field obeys the Dirac equation without assuming the Dirac equation applies, but with the added assumptions of translation invariance and with the aid of the induced 12 to 21 relationship between representations.

The translation dependent parts of the 12-covariant field vector  $\psi_l^{(12)}(x^\mu)$  and the related 21-covariant field vector  $\tilde{\psi}_l^{(21)}(x^\mu)$  depend also on coordinates via translation matrices. This means that the spin components depend on position. The coordinate dependence is just what is needed for the sum of the 4-vector currents of fixed momentum and spin,  $(\bar{u}\gamma^\mu u)^{(12)} + (\tilde{u}\gamma^\mu \tilde{u})^{(21)}$ , to be proportional to the vector potential of the just discussed Dirac equation solution. The evidence for this is that the sum of the currents obeys the Maxwell equations for the Dirac equation solution as the source. A solution to the Maxwell equations appears without assuming the Maxwell equations. However, while sums of Dirac equation solutions with different momenta and spin are still Dirac equation solutions, the currents of sums

of Dirac equation solutions are no longer proportional to the vector potential because the current is quadratic in covariant field vectors. Thus the Maxwell equations solutions found here are limited to one momentum and one spin each.

Section 2 describes the spin 1/2 representations of the Poincaré group of spacetime transformations connected to the identity. There are two, called the 12- and 21-representations. Section 3 discusses the relationship between the two representations, which turn out to be related by parity. The details of the construction of the covariant field vector from canonical field vectors are presented in Section 4, and from the covariant and canonical transformation formulas the formulas for the coefficient functions are derived. The relationship between 12- and 21-covariant fields induced from the relationship between the representations occupies Section 5. As shown in Section 6 the translation invariant parts of the related fields together obey the Dirac equation for free fields. In Section 7, the vector potential for a fixed momentum source from Section 6 is shown to be the sum of the 12- and 21- field currents. Appendix A collects the results obtained by considering the representation adjoint to the canonical unitary representation of the main text, i.e. the construction of the covariant field vector in Appendix A is taken over fields that transform like creation operators. Appendix B contains a problem set. Problems 4, 5, and 6 outline a crude, heuristic physical basis that implies the scale of spin effects due to translations is on the order of one-half of a Compton wavelength.

## 2 Spin 1/2 Poincaré Representations

It is convenient sometimes to work with a definite set of four  $4 \times 4$  gamma matrices  $\gamma^\mu$ ,  $\mu \in \{1, 2, 3, 4\}$ , chosen here to be

$$\gamma^k = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix} \quad \text{and} \quad \gamma^4 = \begin{pmatrix} 0 & \sigma^4 \\ \sigma^4 & 0 \end{pmatrix} \quad , \quad (1)$$

where  $k \in \{1, 2, 3\}$ , ‘0’ stands for the  $2 \times 2$  null matrix, the  $\sigma^k$  are  $2 \times 2$  Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad (2)$$

and  $\sigma^4$  is the  $2 \times 2$  unit matrix. Also define the matrix  $\gamma^5$  by

$$\gamma^5 = -i\gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad (3)$$

where ‘1’ stands for the  $2 \times 2$  unit matrix. The results below that are displayed with the choice of  $\gamma$ s in (1) can be transferred by similarity transformations to other sets of gamma matrices.

By direct calculation from (1) one can verify that the anticommutators of gamma matrices are proportional to the metric,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \quad , \quad (4)$$

where the  $4 \times 4$  unit matrix is understood on the right and  $\eta^{\mu\nu}$  is the metric which is diagonal in this representation,

$$\eta = \text{diag}(-1, -1, -1, +1) \quad . \quad (5)$$

Equation (4) is the defining characteristic of  $4 \times 4$  gamma matrices. [6]

The generators of the Poincaré algebra are the angular momentum  $J^{\mu\nu}$  and momentum  $P^\mu$ . The  $J^{\mu\nu}$  can be represented by the following matrices

$$J^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu] = -\frac{i}{4}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \quad . \quad (6)$$

By (1), one finds that

$$J^{ij} = -\frac{1}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad \text{and} \quad J^{k4} = \frac{i}{2} \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} \quad , \quad (7)$$

where  $\epsilon^{ijk} = (k-i)(k-j)(j-i)/2$  is the antisymmetric symbol defined for  $i, j, k \in \{1, 2, 3\}$ . By (6), the generators  $J^{\mu\nu}$  are antisymmetric in  $\mu\nu$  implying that the four  $J^{\mu\mu}$  are null and of the twelve remaining just six are independent.

The momentum matrices in the ‘12-representation’ are defined by

$$P_{(12)}^\mu = \frac{K^{(12)}}{2}(1 + \gamma^5)\gamma^\mu \quad , \quad (8)$$

where  $K^{(12)}$  is a constant. By (1), one finds that

$$P_{(12)}^k = \begin{pmatrix} 0 & -K^{(12)}\sigma^k \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_{(12)}^4 = \begin{pmatrix} 0 & K^{(12)}\sigma^4 \\ 0 & 0 \end{pmatrix} \quad , \quad (9)$$

where ‘(12)’ designates the nonzero block of the matrix  $P_{(12)}^\mu$ .

Likewise, the momentum matrices in the ‘21-representation’ are defined by

$$P_{(21)}^\mu = \frac{K^{(21)}}{2}(1 - \gamma^5)\gamma^\mu \quad , \quad (10)$$

where  $K^{(21)}$  is a constant. For the choice of  $\gamma$ s in (1), one finds that

$$P_{(21)}^k = \begin{pmatrix} 0 & 0 \\ K^{(21)}\sigma^k & 0 \end{pmatrix} \quad \text{and} \quad P_{(21)}^4 = \begin{pmatrix} 0 & 0 \\ K^{(21)}\sigma^4 & 0 \end{pmatrix} \quad . \quad (11)$$

The 12- and the 21-momentum matrices have special properties that derive from their having just one off-diagonal nonzero block for the  $\gamma$ s in (1). The matrices  $(1 \pm \gamma^5)/2$  in (8) and (10) are unchanged by squaring and are therefore projection operators. By the definition of  $\gamma^5$ , (3), one finds that

$$[\frac{1}{2}(1 + \gamma^5)][\frac{1}{2}(1 - \gamma^5)] = 0 \quad \text{and} \quad [\frac{1}{2}(1 \pm \gamma^5)][\frac{1}{2}(1 \pm \gamma^5)] = \frac{1}{2}(1 \pm \gamma^5) \quad . \quad (12)$$

From this and the momentum matrix definitions (8) and (10) it follows that

$$\frac{1}{2}(1 - \gamma^5)P_{(12)}^\mu = 0 \quad \text{and} \quad \frac{1}{2}(1 + \gamma^5)P_{(21)}^\mu = 0 \quad . \quad (13)$$

These expressions mean that certain parts of any vector are translation matrix independent, as will be shown in a later section.

Collecting generators from (6), (8) and (10) makes two representations of the Poincaré algebra, the 12-representation

$$(J^{\mu\nu}, P_{(12)}^\rho) \quad (14)$$

and the 21-representation

$$(J^{\mu\nu}, P_{(21)}^\rho) \quad . \quad (15)$$

The 12- and 21-representations have angular momentum matrices  $J^{\mu\nu}$  in common, differing by the momentum generators. It is straightforward to show that each representation obeys the commutation rules of the Poincaré algebra, [7]

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} \quad , \quad (16)$$

$$i[P^\mu, J^{\rho\sigma}] = \eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho \quad , \quad (17)$$

and

$$[P^\mu, P^\nu] = 0 \quad . \quad (18)$$

[The commutation rules (16) show that the angular momentum matrices  $J^{\mu\nu}$  represent the homogeneous Lorentz algebra. The spin can be taken as either  $(0, 1/2) \oplus (1/2, 0)$  or  $(1/2, 0) \oplus (0, 1/2)$  since these differ by a similarity transformation.]

Any Poincaré transformation can be written as a Lorentz transformation  $\Lambda$  followed by a translation through some displacement  $b$ . The transformation  $(\Lambda, b)$  depends on a set of antisymmetric parameters  $\omega_{\mu\nu}$  that determine  $\Lambda$  and is represented by the matrix  $D_{\bar{u}}(\Lambda, b)$ , where

$$D_{\bar{u}}^{(12)}(\Lambda, b) = \exp(-ib_\mu P_{(12)}^\mu) \exp(i\omega_{\mu\nu} J^{\mu\nu}/2) \quad , \quad (19)$$

and

$$D_{\bar{u}}^{(21)}(\Lambda, b) = \exp(-ib_\mu P_{(21)}^\mu) \exp(i\omega_{\mu\nu} J^{\mu\nu}/2) \quad , \quad (20)$$

with the two different sets of momentum matrices,  $P_{(12)}^\mu$  and  $P_{(21)}^\mu$ , distinguishing the two different representations. Since the momenta matrices are triangular, all quadratic and higher order terms vanish in the matrix exponent in  $P$ . The two representations agree for homogeneous Lorentz transformations, i.e.  $(\Lambda, 0)$ ,

$$D_{\bar{u}}^{(12)}(\Lambda, 0) = D_{\bar{u}}^{(21)}(\Lambda, 0) \equiv D_{\bar{u}}(\Lambda) \quad . \quad (21)$$

For example, the matrix representing a rotation through an angle  $\xi$  about an axis along the unit vector  $n^i = \{\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta\}$  is given by

$$D(R(\xi \mathbf{n})) = \cos(\xi/2) + i \sin(\xi/2) \gamma^5 \gamma^4 \gamma^j \mathbf{n}_j \quad , \quad (22)$$

where  $\omega_{ij} = \xi \epsilon_{ijk} n^k$  and the  $4 \times 4$  unit matrix is understood in the cosine term.

A boost along  $\mathbf{n}$  taking the rest frame to the velocity  $v\mathbf{n}$  with  $v = \tanh \xi$  is given by

$$D(B(\xi \mathbf{n})) = \cosh(\xi/2) - \sinh(\xi/2) \gamma^4 \gamma^j \mathbf{n}_j \quad (23)$$

where  $\omega_{i4} = -\omega_{4i} = \xi n_i = -\xi n^i$ .

The translation along  $b^\mu$  in the 12-representation is represented by

$$D^{(12)}(1, b^\mu) = 1 - ib_\sigma P_{(12)}^\sigma = 1 - \frac{i}{2} K^{(12)} b_\sigma (1 + \gamma^5) \gamma^\sigma \quad (24)$$

and in the 21-representation by

$$D^{(21)}(1, b^\mu) = 1 - ib_\sigma P_{(21)}^\sigma = 1 - \frac{i}{2} K^{(21)} b_\sigma (1 - \gamma^5) \gamma^\sigma \quad . \quad (25)$$

Applying the projection operators  $(1 \pm \gamma^5)/2$  can nullify the effects of the translation matrices. By (12), (13), (24) and (25), one finds that

$$\frac{1}{2}(1 - \gamma^5) D^{(12)}(\Lambda, b^\mu) = \frac{1}{2}(1 - \gamma^5) D^{(12)}(1, b^\mu) D^{(12)}(\Lambda, 0) = \frac{1}{2}(1 - \gamma^5) D(\Lambda) \quad (26)$$

and

$$\frac{1}{2}(1 + \gamma^5) D^{(21)}(\Lambda, b^\mu) = \frac{1}{2}(1 + \gamma^5) D^{(21)}(1, b^\mu) D^{(12)}(\Lambda, 0) = \frac{1}{2}(1 + \gamma^5) D(\Lambda) \quad , \quad (27)$$

where there is no  $b^\mu$  dependence in either result. These properties have use below.

### 3 Relating the 12- and 21-Representations

The 12- and 21-representations are related. To see this, begin by noting that a similarity transformation with  $\gamma^4$  lowers the 4-vector indices of  $\gamma^\mu$  and, therefore, changes the sign of  $\gamma^5$ ,

$$\gamma^4 \gamma^\mu \gamma^4 = \eta_{\mu\nu} \gamma^\nu = \gamma_\mu \quad \text{and} \quad \gamma^4 \gamma^5 \gamma^4 = -\gamma^5 \quad , \quad (28)$$

which can be verified directly from definitions (1) and (3). By definitions (6), (8), and (10) one finds that

$$\gamma^4 J^{\mu\nu} \gamma^4 = J_{\mu\nu} \quad (29)$$

and

$$\gamma^4 P_{(12)}^\mu \gamma^4 = P_{(21)\mu} \quad \text{and} \quad \gamma^4 P_{(21)}^\mu \gamma^4 = P_{(12)\mu} \quad , \quad (30)$$

where the constants  $K^{(12)}$  and  $K^{(21)}$  have been assumed to be equal,

$$K^{(12)} = K^{(21)} = K \quad . \quad (31)$$

This simplifying assumption means that translations affect spin with the same distance scale in both 12- and 21-representations. (An estimate of the distance scale  $1/K$  is deduced in problems 4, 5, and 6 of Appendix B.)

By (29) and (30), the two sets of generators, (14) and (15), are related by

$$\gamma^4 (J^{\mu\nu}, P_{(12)}^\rho) \gamma^4 = (J_{\mu\nu}, P_{(21)\rho}) \quad \text{and} \quad \gamma^4 (J^{\mu\nu}, P_{(21)}^\rho) \gamma^4 = (J_{\mu\nu}, P_{(12)\rho}) \quad . \quad (32)$$

It is clear that the 12/21 transition from one representation to the other involves a similarity transformation which is equivalent to an exchange of contravariant for covariant indices. The index exchange for the metric here,  $\eta = \text{diag}\{-1, -1, -1, +1\}$ , is equivalent to a parity transformation since all spatial components are replaced with their negatives and time components are unchanged.

The matrix representing a Poincaré transformation  $(\Lambda, b)$  in one representation is similar to a transformation  $(\tilde{\Lambda}, \tilde{b})$  in the other representation. To show this, use (32) which implies that

$$\begin{aligned} \gamma^4 D_{(12)\bar{U}}(\Lambda, b) \gamma^4 &= \exp(-ib_\mu \gamma^4 P_{(12)}^\mu \gamma^4) \exp(i\omega_{\mu\nu} \gamma^4 J^{\mu\nu} \gamma^4 / 2) \\ &= \exp(-ib_\mu \eta_{\rho\mu} P_{(21)}^\rho) \exp(i\omega_{\mu\nu} \eta_{\rho\mu} \eta_{\sigma\nu} J^{\rho\sigma} / 2) \end{aligned} \quad (33)$$

By (19), the expression on the right is the 21-representation of a transformation  $(\tilde{\Lambda}, \tilde{b})$ ,

$$\gamma^4 D_{(12)\bar{U}}(\Lambda, b) \gamma^4 = D_{(21)\bar{U}}(\tilde{\Lambda}, \tilde{b}) \quad , \quad (34)$$



where the parameters  $\tilde{\omega}_{\mu\nu}$  and  $\tilde{b}_\mu$  for  $(\tilde{\Lambda}, \tilde{b})$  are determined by

$$\tilde{\omega}_{\rho\sigma} = \omega_{\mu\nu}\eta_{\rho\mu}\eta_{\sigma\nu} = \omega_{\mu\nu}\eta^{\rho\mu}\eta^{\sigma\nu} = \omega^{\rho\sigma} \quad \text{and} \quad \tilde{b}_\rho = b_\mu\eta_{\rho\mu} = b_\mu\eta^{\rho\mu} = b^\rho \quad . \quad (35)$$

Call this the ‘12/21 transition.’ Clearly, the 12/21 transition exchanges covariant and contravariant indices.

The metric here is  $\eta = \text{diag}\{-1, -1, -1, +1\}$ , so the 12/21 transition changes the sign of parameters with an odd number of spatial indices and preserves the sign of those with an even number of spatial indices. In particular, the identity transformation  $(1, 0)$  is mapped to the identity transformation. Pure rotations are determined by parameters  $\omega^{ij}$  with two spatial indices so the rotation  $R(\theta\mathbf{n})$  is invariant. A pure boost  $B(\phi\hat{\mathbf{p}})$  in the direction  $\hat{p}$  is determined by  $\omega^{i4} = \phi\hat{\mathbf{p}}^i$ . Hence when the transformation  $\Lambda$  is a pure boost along  $\hat{\mathbf{p}}$ , then  $\tilde{\Lambda}$  is the pure boost in the opposite direction,  $B(-\phi\hat{\mathbf{p}})$ . Finally, one can see by (35) that the translation  $T(\{b^k, b^4\})$  corresponds to the translation  $T(\{-b^k, b^4\})$ . Thus for a pure rotation  $R$ , a boost  $B$ , and a translation  $T$  one finds that 12/21 transition determines transformations  $\tilde{R}$ ,  $\tilde{B}$ , and  $\tilde{T}$  given by

$$\begin{aligned} \tilde{R}(\theta\mathbf{n}) &= R(\theta\mathbf{n}) & \tilde{B}(\phi\hat{\mathbf{p}}) &= B(-\phi\hat{\mathbf{p}}) \\ \tilde{T}(\{b^k, b^4\}) &= T(\{-b^k, b^4\}) \quad . \end{aligned} \quad (36)$$

Note that the 12/21 transition,  $(\Lambda, b)$  into  $(\tilde{\Lambda}, \tilde{b})$ , is equivalent to a parity transformation.

Spacial inversion, i.e. the parity operation, changes the relative orientation of the  $x, y, z$  axes and cannot be built from infinitesimal rotations, boosts, or translations since these can not change the axes’ orientation. In this paper parity transformations are not represented, only those transformations arising from infinitesimal transformations connected to the identity are discussed. Thus, in the 12-representation, only Poincaré transformations connected to the identity are considered. The same is true for the 21-representation. Spacial inversion, i.e. parity, arises only when comparing the 12- and 21-representations.

## 4 Invariant Coefficient Hypothesis

The calculation presented in this section closely follows Weinberg, [1].

With invariant coefficient functions  $u$ , the construction  $\psi = \sum u a$  transforms with a Poincaré transformation  $(\Lambda, b)$  according to

$$\psi = \sum_a u a \quad \rightarrow \quad \psi' = \sum_{a'} u a' \quad . \quad (37)$$

If the coefficient functions are allowed to change freely, i.e.  $u'$  can differ from  $u$  and possibly depend on the transformation, then no new information is gained; the transformed equation

on the right above would then merely reflect the existence of the original on the left. Having invariant, nonzero coefficient functions,  $u' = u \neq 0$  constrains the coefficient functions, leading to constraints on  $\psi$ . All the results below are based on the constraints imposed by the Invariant Coefficient Hypothesis and the transformation characters of the covariant field vector  $\psi$  and the canonical field vectors  $a$ .

The covariant vector fields  $\psi^{(12)}$  and  $\psi^{(21)}$ , are required to be linear combinations of canonical field vectors  $a$ ,

$$\psi_l^{(12)}(x) = \sum_{\sigma} \int d^3p \, u_l^{(12)}(x; \mathbf{p}, \sigma) a(\mathbf{p}, \sigma) \quad , \quad (38)$$

and

$$\psi_l^{(21)}(x) = \sum_{\sigma} \int d^3p \, u_l^{(21)}(x; \mathbf{p}, \sigma) a(\mathbf{p}, \sigma) \quad , \quad (39)$$

The symbol  $\mathbf{p}$  denotes the space components of the momentum,  $\{p^x, p^y, p^z\}$ . The spacial components  $\mathbf{p}$  determine the time component because the mass is held fixed and  $p^t$  is required to be positive for this class of Poincaré transformations.

The covariant fields  $\psi^{(12)}$  and  $\psi^{(21)}$  transform like relativistic field operators, [8], [9]

$$U(\Lambda, b) \psi_l^{(12)}(x) U^{-1}(\Lambda, b) = \sum_{\bar{l}} D_{\bar{l}l}^{(12)-1}(\Lambda, b) \psi_{\bar{l}}^{(12)}(\Lambda x + b) \quad , \quad (40)$$

and

$$U(\Lambda, b) \psi_l^{(21)}(x) U^{-1}(\Lambda, b) = \sum_{\bar{l}} D_{\bar{l}l}^{(21)-1}(\Lambda, b) \psi_{\bar{l}}^{(21)}(\Lambda x + b) \quad , \quad (41)$$

where  $D^{(12)}(\Lambda, b)$  and  $D^{(21)}(\Lambda, b)$  are the spin 1/2 covariant nonunitary matrices representing the spacetime transformation  $(\Lambda, b)$  in the 12- and 21-representations of the Poincaré group discussed in Section 2 above. The matrices transform the components labeled by the index  $l$  and there is a differential representation that transforms the field defined on the space of continuous variables  $x \rightarrow \Lambda x + b$ .

The canonical field vectors  $a$  transform like annihilation operators and single particle states, [10], [11]

$$U(\Lambda, b) a(\mathbf{p}, \sigma) U^{-1}(\Lambda, b) = e^{-i\Lambda p \cdot b} \sqrt{\frac{(\Lambda p)^t}{p^t}} \sum_{\bar{\sigma}} D_{\bar{\sigma}\sigma}^{(j)}(W^{-1}(\Lambda, p)) a(\mathbf{p}_{\Lambda}, \bar{\sigma}) \quad , \quad (42)$$

where  $j$  is the spin of the particle and where

$$W(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p) \quad , \quad (43)$$

with  $L(p)$  a standard transformation taking the standard 4-vector  $k^\mu = \{0, 0, 0, M\}$  to  $p$ , e.g. a rotation taking  $\hat{p}$  to  $\hat{z}$  followed by a boost along  $z$  followed by a rotation taking the unit vector  $\hat{z}$  to  $\hat{p}$ . The momenta are restricted by

$$p_\mu p^\mu = M^2 \quad , \quad (44)$$

where  $M$  is constant. Thus  $p^\mu$  is determined by  $\mathbf{p}$  together with  $p^4 > 0$ . The space components of the transformed momentum  $\Lambda p$  are denoted  $\mathbf{p}_\Lambda$ . Since  $W(\Lambda, p)k = k$ , it follows that  $W$  is a rotation. The matrices  $D^{(j)}$  form a unitary spin  $j$  representation of rotations.

The dependence of  $u_l^{(12)}(x; \mathbf{p}, \sigma)$  on coordinates  $x$  and translation  $b$  can be taken care of by defining  $u_{\bar{l}}(\mathbf{p}, \sigma)$  in the following expressions for the coefficient functions,

$$u_l^{(12)}(x; \mathbf{p}, \sigma) = (2\pi)^{-3/2} e^{-ip \cdot x} \sum_{\bar{l}} D_{\bar{l}l}^{(12)}(1, x) u_{\bar{l}}(\mathbf{p}, \sigma) \quad (45)$$

and

$$u_l^{(21)}(x; \mathbf{p}, \sigma) = (2\pi)^{-3/2} e^{-ip \cdot x} \sum_{\bar{l}} D_{\bar{l}l}^{(21)}(1, x) u_{\bar{l}}(\mathbf{p}, \sigma) \quad . \quad (46)$$

By (40), (42), (45) and (46), the transformed equation  $\psi' = \sum_{a'} u a'$  reduces to

$$\sum_{\bar{l}} D_{\bar{l}l}(\Lambda) u_{\bar{l}}(\mathbf{p}, \sigma) = \sqrt{\frac{(\Lambda p)^t}{p^t}} \sum_{\bar{\sigma}} u_l(\mathbf{p}_\Lambda, \bar{\sigma}) D_{\bar{\sigma}\sigma}^{(j)}(W(\Lambda, p)) \quad . \quad (47)$$

The labels (12) and (21) are dropped on  $u_{\bar{l}}(\mathbf{p}, \sigma)$  because equation (47) for  $u_{\bar{l}}(\mathbf{p}, \sigma)$  is the same equation in both the 12- and the 21-representations. Of course if there is more than one solution to the equation, then the function  $u_{\bar{l}}(\mathbf{p}, \sigma)$  for the 12-representation may differ from the function  $u_{\bar{l}}(\mathbf{p}, \sigma)$  for the 21-representation. In fact, as shown below, equation (47) determines  $u_{\bar{l}}(\mathbf{p}, \sigma)$  in terms of two arbitrary constants and the constants can be different for the 12- and 21-representations.

To determine the particle spin  $j$ , consider the case when  $p^\mu$  is the standard 4-vector,  $p^\mu = k^\mu$ , which has zero spatial momentum components  $\mathbf{p} = \mathbf{k} = 0$  and let  $\Lambda$  be a rotation  $R$ . The rotation has no effect on the null spacial components of  $k^\mu$ , and it follows that  $Rk = k$  and  $\mathbf{p}_R = 0$ . Also,  $W(R, k) = L^{-1}(Rk)RL(k) = R$  because  $L(k) = L^{-1}(k) = 1$ . In this case (47) reads

$$\sum_{\bar{l}} D_{\bar{l}l}(R) u_{\bar{l}}(0, \sigma) = \sum_{\bar{\sigma}} u_l(0, \bar{\sigma}) D_{\bar{\sigma}\sigma}^{(j)}(R) \quad . \quad (48)$$

By (7) and (19) with  $\Lambda = R$  and  $b = 0$ , the matrix  $D_{\bar{l}l}(R)$  has a block diagonal form and (48) implies that

$$\frac{1}{2} \sum_{\bar{m}} \sigma_{m\bar{m}}^k u_{\bar{m}\pm}(0, \sigma) = \sum_{\bar{\sigma}} u_{m\pm}(0, \bar{\sigma}) J_{\bar{\sigma}\sigma}^{(j)k} \quad , \quad (49)$$

where  $J^{(j)k}$  is an angular momentum matrix for the canonical representation, generating rotations not boosts. By one of Schur's lemmas [12] it follows that, unless the coefficient functions vanish,  $j = 1/2$  and the generators  $\sigma^k/2$  and  $J^{(j)k}$  are similar. They may be taken to be identical,

$$J^{(j)k} = \frac{1}{2}\sigma^k \quad . \quad (50)$$

Knowing the generators  $J^{(j)k}$  determines the representation  $D^{(j)}$  which can now be used to determine  $u(0, \sigma)$ . Replacing  $J^{(j)k}$  in (49) with  $\sigma^k/2$  implies that the coefficients  $u_{m\pm}(0, \sigma)$  form matrices, one for  $+$  and one for  $-$ , that commute with the Pauli matrices  $\sigma^k$  and are therefore proportional to the unit matrix,  $u_{m\pm}(0, \sigma) = c_{\pm}\delta_{m\sigma}$ ,

$$u_l(0, +1/2) = \begin{pmatrix} u_{m+}(0, +1/2) \\ u_{m-}(0, +1/2) \end{pmatrix} = \begin{pmatrix} c_+\delta_{m,+1/2} \\ c_-\delta_{m,+1/2} \end{pmatrix} = \begin{pmatrix} c_+ \\ 0 \\ c_- \\ 0 \end{pmatrix} \quad (51)$$

and

$$u_l(0, -1/2) = \begin{pmatrix} u_{m+}(0, -1/2) \\ u_{m-}(0, -1/2) \end{pmatrix} = \begin{pmatrix} c_+\delta_{m,-1/2} \\ c_-\delta_{m,-1/2} \end{pmatrix} = \begin{pmatrix} 0 \\ c_+ \\ 0 \\ c_- \end{pmatrix} \quad . \quad (52)$$

Thus the coefficient functions for  $p^\mu = k^\mu$  are determined by two parameters  $c_{\pm}$ . These may be different in the 12- and 21-representations.

To find the  $u(\mathbf{p}, \sigma)$  in (45) consider another case of (47), this time when  $\Lambda = L^{-1}(p)$ . Since  $L(p)$  takes  $k$  to  $p$ , it follows that  $L^{-1}(p)$  takes  $p$  to  $k$  and that  $W(L^{-1}(p), p) = L^{-1}(L^{-1}(p)p)L^{-1}(p)L(p) = L^{-1}(k) = 1$ . Now (47) becomes, for this case,

$$u_l(\mathbf{p}, \sigma) = \sqrt{\frac{M}{p^t}} \sum_{\bar{l}} D_{l\bar{l}}(L(p)) u_{\bar{l}}(0, \sigma) \quad . \quad (53)$$

By (45), (46), (51), and (53) the coefficient functions  $u_l^{(12)}(x; \mathbf{p}, \sigma)$  and  $u_l^{(21)}(x; \mathbf{p}, \sigma)$  are given by

$$u_l^{(12)}(x; \mathbf{p}, \sigma) = D_{l\bar{l}}^{(12)}(1, x) u_{\bar{l}}(x; \mathbf{p}, \sigma) \quad (54)$$

$$u_l^{(21)}(x; \mathbf{p}, \sigma) = D_{l\bar{l}}^{(21)}(1, x) u_{\bar{l}}(x; \mathbf{p}, \sigma) \quad , \quad (55)$$

where  $u(x; \mathbf{p}, \sigma)$  is given by

$$u_{\bar{l}}(x; \mathbf{p}, \sigma) = (2\pi)^{-3/2} \sqrt{\frac{M}{p^t}} e^{-ip \cdot x} \sum_n D_{\bar{l}n}(L(p)) u_n(0, \sigma) \quad , \quad (56)$$

for both the 12- and 21-representations. The constants  $c_{\pm}$  in  $u_n(0, \sigma)$ , (51) and (52), may be different for the 12- and 21-representations. Therefore the coefficient functions  $u_l^{(12)}(x; \mathbf{p}, \sigma)$  are determined by the parameters  $c_{\pm}$  and the coefficient functions  $u_l^{(21)}(x; \mathbf{p}, \sigma)$  are determined by the generally different parameters  $c''_{\pm}$ .

The structure of the expressions in (54), (55) and (56) reflect the observation that the 12- and 21- representations agree for rotations and boosts, i.e. note the  $D(L(p))$  in the expression (56) for  $u(x; \mathbf{p}, \sigma)$ , but the representations differ for translations, i.e. note the  $D^{(12)}(1, x)$  and  $D^{(21)}(1, x)$  in (54) and (55) that distinguish  $u^{(12)}(x; \mathbf{p}, \sigma)$  from  $u^{(21)}(x; \mathbf{p}, \sigma)$ .

## 5 Relating the 12- and 21-Fields

As discussed in Section 3, the 12- and 21-representations of the Poincaré group of spacetime transformations are related by a similarity transformation in the equivalent form of an exchange of contravariant and covariant indices as displayed in (34) and (35). This relationship induces a relationship between the 12-field  $\psi^{(12)}$  and the 21-field  $\psi^{(21)}$ .

By (34), (35), (54), and (56), it follows that a 12- coefficient function determines a 21- coefficient function,

$$\gamma^4 u^{(12)}(x, \mathbf{p}, \sigma) = (2\pi)^{-3/2} \sqrt{\frac{M}{p^t}} e^{-ip \cdot x} \gamma^4 D^{(12)}(L(p), x) \gamma^4 \gamma^4 u(0, \sigma) \quad (57)$$

$$\gamma^4 u^{(12)}(x, \mathbf{p}, \sigma) = \tilde{u}^{(21)}(\tilde{x}, \tilde{\mathbf{p}}, \sigma) \quad ,$$

where

$$\tilde{u}^{(21)}(\tilde{x}, \tilde{\mathbf{p}}, \sigma) = (2\pi)^{-3/2} \sqrt{\frac{M}{\tilde{p}^t}} e^{-i\tilde{p} \cdot \tilde{x}} D^{(21)}(L(\tilde{p}), \tilde{x}) \tilde{u}(0, \sigma) \quad . \quad (58)$$

In these equations,

$$\tilde{p}^{\mu} = \eta_{\mu\nu} p^{\nu} = p_{\mu} \quad \text{and} \quad \tilde{x}^{\mu} = x_{\mu} \quad (59)$$

and

$$\tilde{u}(0, \sigma) = \gamma^4 u(0, \sigma) \quad . \quad (60)$$

For  $\sigma = +1/2$  and the  $\gamma$ s in (1), equation (60) becomes

$$\tilde{u}(0, 1/2) = \begin{pmatrix} \tilde{c}_+ \\ 0 \\ \tilde{c}_- \\ 0 \end{pmatrix} = \gamma^4 u(0, 1/2) = \begin{pmatrix} c_- \\ 0 \\ c_+ \\ 0 \end{pmatrix} \quad . \quad (61)$$

and, for  $\sigma = -1/2$ ,

$$\tilde{u}(0, -1/2) = \begin{pmatrix} 0 \\ \tilde{c}_+ \\ 0 \\ \tilde{c}_- \end{pmatrix} = \gamma^4 u(0, -1/2) = \begin{pmatrix} 0 \\ c_- \\ 0 \\ c_+ \end{pmatrix} . \quad (62)$$

Thus the parameters  $c_\pm$  for  $u(0, \sigma)$  and the parameters  $\tilde{c}_\pm$  for  $\tilde{u}(0, \sigma)$ , see (51) and (52), satisfy

$$\tilde{c}_\pm = c_\mp . \quad (63)$$

The induced relationship (57) involves opposite momenta,  $\mathbf{p}$  and  $\tilde{\mathbf{p}} = -\mathbf{p}$ . To find the relationship at equal momenta, go through zero momentum, using  $\tilde{u}(0, \sigma)$  and  $u(0, \sigma)$ . With parameters  $\tilde{c}_\pm = c_\mp$ , by (63), it follows from (56) and its equivalent for  $\tilde{u}$  that

$$\tilde{u}_l(x; \mathbf{p}, \sigma, \tilde{c}_+, \tilde{c}_-) = u_l(x; \mathbf{p}, \sigma, c_-, c_+) , \quad (64)$$

where the dependence on the parameters  $c_\pm$  is displayed explicitly. Thus the relationship between the 12- and 21-representations selects a 21-coefficient function  $\tilde{u}$  that differs from the 12-coefficient function  $u$  by an exchange of parameters  $c_+ \leftrightarrow c_-$  at any momentum  $\mathbf{p}$ . Furthermore, applying the transition twice brings back the original situation.

## 6 Translation Matrix Invariance; Dirac's Equation

The 12- and 21-representations described in Section 2 have nontrivial momentum matrices. Thus, by (40) and (41), neither  $\psi^{(12)}$  nor  $\tilde{\psi}^{(21)}$  are invariant under translations  $b$  in  $D(\Lambda, b)$ . However, the momentum matrices have special properties. For example, when written with the  $\gamma$ s in (1), the momentum matrix  $P_{(12)}^\mu$  in (9) has a null second row. This means that the corresponding components of a coefficient function  $u_l$ , say, are unchanged upon multiplication by a translation matrix,  $1 - ib_\mu P_{(12)}^\mu$ . These components but not the others are invariant under matrix translations. While the results do not depend on the choice of  $\gamma$ s, from expressions (9) and (11) for the momenta which show the null rows, it follows that translation matrix invariant expressions can be found.

To obtain such an invariant which has a full complement of not-necessarily-null components it is necessary to mix parts from both and the 12- and 21-representations, consider defining the 'mixed' quantity  $\psi^{(21/12)}$  ,

$$\psi^{(21/12)} = \frac{\alpha}{2}(1 - \gamma^5)\psi^{(12)} + \frac{\beta}{2}(1 + \gamma^5)\tilde{\psi}^{(21)} , \quad (65)$$

where  $\alpha$  and  $\beta$  are arbitrary complex constants.

The following calculation shows that  $\psi^{(21/12)}$  is indeed invariant to matrix translation:

$$\begin{aligned}
U(\Lambda, b)\psi_l^{(21/12)}(x)U^{-1}(\Lambda, b) &= \\
&= \frac{\alpha}{2}(1 - \gamma^5)U(\Lambda, b)\psi_l^{(12)}(x)U^{-1}(\Lambda, b) + \frac{\beta}{2}(1 + \gamma^5)U(\Lambda, b)\tilde{\psi}_l^{(21)}(x)U^{-1}(\Lambda, b) \\
&= \frac{\alpha}{2}(1 - \gamma^5)D^{(12)}_{\bar{u}}^{-1}(\Lambda, b)\psi_l^{(12)}(\Lambda x + b) + \\
&\quad + \frac{\beta}{2}(1 + \gamma^5)D^{(21)}_{\bar{u}}^{-1}(\Lambda, b)\tilde{\psi}_l^{(21)}(\Lambda x + b) \quad ,
\end{aligned}$$

which implies by (26) and (27) that

$$U(\Lambda, b)\psi_l^{(21/12)}(x)U^{-1}(\Lambda, b) = D_{\bar{u}}^{-1}(\Lambda)\psi_l^{(21/12)}(\Lambda x + b) \quad , \quad (66)$$

where the absence of  $b_\mu$  in  $D^{-1}(\Lambda)$  shows the invariance under matrix translations. Compare this with the  $D^{-1}(\Lambda, b)$  in (40) and (41).

In (66), the differential representation of translations is evident in the dependence of  $x$ ,  $x \rightarrow \Lambda x + b$ . But the matrix  $D^{-1}(\Lambda)$  on the right above indicates that the matrix shuffling of the components of  $\psi_l^{(21/12)}$  is independent of  $b$ , so  $\psi_l^{(21/12)}$  is invariant under *matrix* translations.

The choice of constants  $\alpha$  and  $\beta$  that most simply determines the Dirac equation is the following,

$$\alpha = \beta = 1, \quad (67)$$

that is, the 12- and 21-fields are equally weighted. The choice of unity is made for convenience. Equation (67) or something like it is essential to obtain the Dirac equation. In Appendix A a different choice ( $\alpha = -\beta = +1$ ) is made for vectors that transform like creation operators in order to arrive at the same Dirac equation as found in this section below. Note that, for  $\alpha = \beta$ , the 12/21 transition leaves  $\psi^{(21/12)}$  unchanged.

Define the ‘mixed coefficient function’  $u^{(21/12)}$  by

$$u_l^{(21/12)}(x; \mathbf{p}, \sigma) = \frac{1}{2}(1 - \gamma^5)u_l^{(12)}(x; \mathbf{p}, \sigma, c_+, c_-) + \frac{1}{2}(1 + \gamma^5)\tilde{u}_l^{(21)}(x; \mathbf{p}, \sigma, \tilde{c}_+, \tilde{c}_-) \quad (68)$$

or for the  $\gamma$ s in (1) and the induced parameter relation  $\tilde{c}_\pm = c_\mp$ , (63), one finds that

$$u_l^{(21/12)}(x; \mathbf{p}, \sigma) = \begin{pmatrix} \tilde{u}_{m+}(x; \mathbf{p}, \sigma, c_-, c_+) \\ u_{m-}(x; \mathbf{p}, \sigma, c_+, c_-) \end{pmatrix} \quad . \quad (69)$$

Expression (69) explains the notation  $(21/12)$  which can be read as ‘21 over 12,’ indicating the origin of the components in the 21- and 12-representations, respectively.

By (54), (58) and (69), one finds that

$$u_l^{(21/12)}(x; \mathbf{p}, \sigma) = (2\pi)^{-3/2} \sqrt{\frac{M}{p^t}} e^{-ip \cdot x} \sum_n D_{\bar{l}n}(L(p)) u_n^{(21/12)}(0, \sigma) \quad , \quad (70)$$

where for the  $\gamma$ s in (1) and by (51), (52), and (63)

$$u_l^{(21/12)}(0, 1/2) = \begin{pmatrix} c_- \\ 0 \\ c_- \\ 0 \end{pmatrix} \quad \text{and} \quad u_l^{(21/12)}(0, -1/2) = \begin{pmatrix} 0 \\ c_- \\ 0 \\ c_- \end{pmatrix} \quad . \quad (71)$$

Thus the quantity  $u^{(21/12)}(0, \sigma)$  is an eigenvector of  $\gamma^4$ ,

$$\gamma^4 u_l^{(21/12)}(0, \sigma) = u_l^{(21/12)}(0, \sigma) \quad . \quad (72)$$

This expression and the expression (70) form the basis of the Dirac equation for massive free spin 1/2 particles.

By (70) and (72), the mixed coefficient function  $u^{(21/12)}$  satisfies

$$D(L(p)) \gamma^4 D^{-1}(L(p)) u_l^{(21/12)}(x; \mathbf{p}, \sigma) = u_l^{(21/12)}(x; \mathbf{p}, \sigma) \quad . \quad (73)$$

For momentum  $p^\mu$  parameterized by

$$p^\mu = M \{ \sinh \xi \cos \phi \sin \theta, \sinh \xi \sin \phi \sin \theta, \sinh \xi \cos \theta, \cosh \xi \} \quad , \quad (74)$$

the matrix representing the special transformation taking  $k$  to  $p$  is given by

$$D(L(p)) = D(R(\phi \hat{\mathbf{z}})) D(R(\theta \hat{\mathbf{y}})) D(B(\xi \hat{\mathbf{z}})) D(R(-\theta \hat{\mathbf{y}})) D(R(-\phi \hat{\mathbf{z}})) \quad . \quad (74.5)$$

To see how the expression  $D(L(p)) \gamma^4 D^{-1}(L(p))$  can be rewritten, consider the case for  $\theta = 0$ , i.e. momentum  $\mathbf{p}$  in the  $\hat{z}$  direction,

$$\begin{aligned} D(B(\xi \hat{\mathbf{z}})) \gamma^4 D^{-1}(B(\xi \hat{\mathbf{z}})) &= \\ &= (\cosh(\xi/2) + \sinh(\xi/2) \gamma^4 \gamma^3) \gamma^4 (\cosh(\xi/2) - \sinh(\xi/2) \gamma^4 \gamma^3) \\ &= (\cosh^2(\xi/2) + \sinh^2(\xi/2)) \gamma^4 - 2 \sinh(\xi/2) \cosh(\xi/2) \gamma^3 \\ &= \cosh(\xi) \gamma^4 - \sinh(\xi) \gamma^3 \quad , \end{aligned} \quad (75)$$



which is just the result of transforming  $\{0, 0, 0, \gamma^4\}$  by  $B^{-1}(\xi\hat{\mathbf{z}})$ . The general rule is that a vector matrix transforms as a second rank tensor and as a vector, i.e.

$$D(\Lambda)\gamma^\mu D^{-1}(\Lambda) = (\Lambda^{-1})^\mu_\nu \gamma^\nu \quad . \quad (76)$$

One can show that  $L^{-1}(p)$  takes  $k^\mu/M = \{0, 0, 0, 1\}$  to  $p_\mu/M$  and  $\gamma^4$  is the fourth component of  $\gamma^\mu$ , it follows from (73) and (76) that

$$\gamma^\mu p_\mu u_l^{(21/12)}(x; \mathbf{p}, \sigma) = M u_l^{(21/12)}(x; \mathbf{p}, \sigma) \quad . \quad (77)$$

By (70) the mixed coefficient functions  $u^{(21/12)}(x; \mathbf{p}, \sigma)$  depend on coordinates  $x^\mu$  only in the plane wave factor  $\exp(-ip \cdot x)$ . It follows that the usual gradient operator  $i\partial_\mu$  can replace the momentum  $p_\mu$  in (77) and  $\psi^{(21/12)}(x)$  must obey the Dirac equation,

$$i\gamma^\mu \partial_\mu \psi_l^{(21/12)}(x) = M \psi_l^{(21/12)}(x) \quad , \quad (78)$$

where  $\partial_\mu = \partial/\partial x^\mu$ . Note that the properties of the canonical vectors  $a(\mathbf{p}, \sigma)$  are irrelevant and the equation for  $\psi^{(21/12)}$  follows from the equation for the coefficients  $u^{(21/12)}$ . This is because the Dirac equation is linear and the canonical vectors  $a(\mathbf{p}, \sigma)$  do not depend on coordinates  $x^\mu$ .

Thus it has been shown in this section that the Dirac equation follows automatically from the Invariant Coefficient Hypothesis for the translation matrix invariant parts of fields that are related simply by the 12/21 transition.

## 7 Current as Vector Potential; Maxwell's Equations

The covariant fields  $\psi$  transform by two types of translation representations, differential and matrix. This is reflected in the coefficient functions  $u^{(12)}(x; \mathbf{p}, \sigma)$  and  $\tilde{u}^{(21)}(x; \mathbf{p}, \sigma)$ , (54) and (55), which depend on coordinates  $x^\mu$  through a plane wave factor,  $\exp(-ip \cdot x)$ , and through matrix translation factors,  $D^{(12)}(1, x)$  and  $D^{(21)}(1, x)$ . The plane wave factor is a spin independent factor, while the translation matrices adjust the spin as a function of position.

In this section, one consequence of the position dependent spin is described. It is shown that the currents  $j^{(12)}$  and  $j^{(21)}$  of these coefficient functions have position dependence such that the sum of the currents,  $j^{(12)\mu} + j^{(21)\mu}$ , has the same position dependence as the vector potential of the current for the coefficient function  $u^{(21/12)}$  discussed in Section 6.

Define the currents  $j^{(12)\mu}$  and  $\tilde{j}^{(21)\mu}$  by

$$j^{(12)\mu}(x; \mathbf{p}, \sigma) = \frac{p^\mu}{M} \bar{u}^{(12)}(x; \mathbf{p}, \sigma) \gamma^\mu u^{(12)}(x; \mathbf{p}, \sigma) \quad (79)$$

and

$$\tilde{j}^{(21)\mu}(x; \mathbf{p}, \sigma) = \frac{p^t}{M} \tilde{u}^{(21)}(x; \mathbf{p}, \sigma) \gamma^\mu \tilde{u}^{(21)}(x; \mathbf{p}, \sigma) \quad , \quad (80)$$

where  $\bar{u} = u^\dagger \gamma^4$ . The current for  $p^\mu$  transforms to the current for  $q^\mu$  by a Lorentz transformation obtained by applying the standard transformation taking  $p^\mu$  to the standard 4-vector  $k^\mu$  followed by the standard transformation from  $k^\mu$  to  $q^\mu$ ,

$$j^{(12)\mu}(x; \mathbf{q}, \sigma) = [L(q)L^{-1}(p)]^\mu_\nu j^{(12)\nu}(L(p)L^{-1}(q)x; \mathbf{p}, \sigma) \quad (80.5)$$

and similarly for  $\tilde{j}^{(21)\mu}$ . Even when  $p^\mu$  and  $q^\mu$  differ by a simple rotation, the transformation  $L(q)L^{-1}(p)$  is not a rotation. These currents do not transform simply.

The plane wave coordinate dependence  $\exp(-ip \cdot x)$  cancels out in  $j^{(12)\mu}$  and  $\tilde{j}^{(21)\mu}$ , so the only dependence on coordinates is in the translation matrices  $D^{(12)}(1, x)$  and  $D^{(21)}(1, x)$  and their adjoints,

$$j^{(12)\mu}(x; \mathbf{p}, \sigma) = \frac{p^t}{M} u^\dagger(0; \mathbf{p}, \sigma) D^{(12)\dagger}(1, x) \gamma^4 \gamma^\mu D^{(12)}(1, x) u(0; \mathbf{p}, \sigma) \quad (81)$$

and

$$\tilde{j}^{(21)\mu}(x; \mathbf{p}, \sigma) = \frac{p^t}{M} \tilde{u}^\dagger(0; \mathbf{p}, \sigma) D^{(21)\dagger}(1, x) \gamma^4 \gamma^\mu D^{(21)}(1, x) \tilde{u}(0; \mathbf{p}, \sigma) \quad . \quad (82)$$

By expressions (24) and (25) for  $D^{(12)}(1, x)$  and  $D^{(21)}(1, x)$ , one sees that  $D^{(12)}(1, x) = 1 - ix_\mu P_{(12)}^\mu$  and  $D^{(21)}(1, x) = 1 - ix_\mu P_{(21)}^\mu$ , i.e. they are linear in the coordinates. Thus the currents are quadratic in the coordinates  $x^\mu$ . In particular, third order partial derivatives vanish and second order partials are constants.

By (24), (25), (81) and (82), the second order partial derivatives of the currents with respect to coordinates  $x^\tau$  and  $x^\kappa$  are found to be

$$\frac{\partial^2}{\partial x^\tau \partial x^\kappa} j^{(12)\mu}(x) = K^2 (\eta_{\alpha\tau} \eta_{\beta\kappa} + \eta_{\alpha\kappa} \eta_{\beta\tau}) \frac{p^t}{M} \bar{u}_-(0; \mathbf{p}, \sigma) \gamma^\alpha \gamma^\mu \gamma^\beta u_-(0; \mathbf{p}, \sigma) \quad (83)$$

and

$$\frac{\partial^2}{\partial x^\tau \partial x^\kappa} \tilde{j}^{(21)\mu}(x) = K^2 (\eta_{\alpha\tau} \eta_{\beta\kappa} + \eta_{\alpha\kappa} \eta_{\beta\tau}) \frac{p^t}{M} \tilde{\bar{u}}_+(0; \mathbf{p}, \sigma) \gamma^\alpha \gamma^\mu \gamma^\beta \tilde{u}_+(0; \mathbf{p}, \sigma) \quad , \quad (84)$$

where, for the  $\gamma$ s in (1),

$$u_-(0; \mathbf{p}, \sigma) \equiv \frac{1}{2} (1 - \gamma^5) u(0; \mathbf{p}, \sigma) = \begin{pmatrix} 0 \\ u_{m-}(0; \mathbf{p}, \sigma) \end{pmatrix} \quad (85)$$

and

$$\tilde{u}_+(0; \mathbf{p}, \sigma) \equiv \frac{1}{2} (1 + \gamma^5) \tilde{u}(0; \mathbf{p}, \sigma) = \begin{pmatrix} \tilde{u}_{m+}(0; \mathbf{p}, \sigma) \\ 0 \end{pmatrix} \quad . \quad (86)$$

By a straightforward calculation, expressions (83) and (84) imply that

$$\partial^\tau \partial_\tau j^{(12)\mu}(x; \mathbf{p}, \sigma) - \partial^\mu \partial_\kappa j^{(12)\kappa}(x; \mathbf{p}, \sigma) = -12K^2 \frac{p^t}{M} \bar{u}_-(x; \mathbf{p}, \sigma) \gamma^\mu u_-(x; \mathbf{p}, \sigma) \quad (87)$$

and

$$\partial^\tau \partial_\tau \tilde{j}^{(21)\mu}(x; \mathbf{p}, \sigma) - \partial^\mu \partial_\kappa \tilde{j}^{(21)\kappa}(x; \mathbf{p}, \sigma) = -12K^2 \frac{p^t}{M} \bar{\tilde{u}}_+(x; \mathbf{p}, \sigma) \gamma^\mu \tilde{u}_+(x; \mathbf{p}, \sigma) \quad , \quad (88)$$

where  $\partial^\tau = \eta^{\tau\nu} \partial / \partial x^\nu$ .

Note that the  $x$ -independent currents of  $u_-(0; \mathbf{p}, \sigma)$  and  $\tilde{u}_+(0; \mathbf{p}, \sigma)$  are the same as the currents of  $u_-(x; \mathbf{p}, \sigma)$  and  $\tilde{u}_+(x; \mathbf{p}, \sigma)$  because these coefficient functions are invariant under matrix translations and depend on position  $x^\mu$  only in a plane wave factor  $\exp(-ip \cdot x)$  that cancels out of the current. And it is just these matrix translation invariant quantities that make up the mixed coefficient function  $u^{(21/12)}$ .

Define the quantity  $a^\mu$  as proportional to the sum of the currents,

$$a^\mu(x; \mathbf{p}, \sigma) = \frac{-q}{12K^2} [j^{(12)\mu}(x; \mathbf{p}, \sigma) + \tilde{j}^{(21)\mu}(x; \mathbf{p}, \sigma)] \quad , \quad (89)$$

where the constant  $q$  is introduced to put the following equations in a familiar form. By (87), (88) and (89) it follows that  $a^\mu$  is a vector potential for the current  $qj^{(21/12)\mu}$  due to  $u^{21/12}$ , i.e.

$$\partial^\tau \partial_\tau a^\mu(x; \mathbf{p}, \sigma) - \partial^\mu \partial_\kappa a^\kappa(x; \mathbf{p}, \sigma) = qj^{(21/12)\mu}(\mathbf{p}, \sigma) \quad , \quad (90)$$

where

$$j^{(21/12)\mu}(\mathbf{p}, \sigma) = \frac{p^t}{M} \bar{u}^{(21/12)}(x; \mathbf{p}, \sigma) \gamma^\mu u^{(21/12)}(x; \mathbf{p}, \sigma) \quad . \quad (91)$$

Note that  $j^{(21/12)\mu}$  does not depend on  $x^\mu$  because  $u^{(21/12)}$  depends on  $x^\mu$  in a phase factor and the phase factors of  $u^{(21/12)}$  and  $\bar{u}^{(21/12)}$  cancel. Thus the current  $j^{(21/12)\mu}$  is constant throughout spacetime. Unlike  $j^{(12)\mu}$ ,  $\tilde{j}^{(21)\mu}$ , and  $a^\mu$  which do not transform simply as the momentum changes, the current  $j^{(21/12)\mu}$  is proportional to the 4-vector momentum  $p^\mu$  and thus the current  $j^{(21/12)\mu}$  transforms as a 4-vector just as  $p^\mu$  does.

Equation (90) shows that  $a^\mu$  is the vector potential because  $a^\mu$  satisfies the Maxwell equation and that defines the vector potentials of the current  $j^{(21/12)\mu}$ . (Other vector potentials satisfy the equation and are related to  $a^\mu$  by gauge transformations.)

To obtain expressions for the associated electromagnetic field, define the quantity  $F^{\mu\nu}$  by

$$F^{\mu\nu} = a^{\mu,\nu} - a^{\nu,\mu} \quad , \quad (92)$$

where the commas denote partial differentiation,

$$a^{\mu,\nu} = \eta^{\nu\sigma} \frac{\partial a^\mu}{\partial x^\sigma} \quad . \quad (93)$$

One of the Maxwell equations is automatically satisfied by the definition (92),

$$F^{\mu\nu,\lambda} + F^{\nu\lambda,\mu} + F^{\lambda\mu,\nu} = 0 \quad . \quad (94)$$

By (89) one finds that

$$\begin{aligned} F^{\mu\nu}(x; \mathbf{p}, \sigma) = & \frac{-q}{3} \left[ \frac{p^t}{(2\pi)^3 K M} \bar{u}(0; \mathbf{p}, \sigma) J^{\mu\nu} u(0; \mathbf{p}, \sigma) + \frac{p^t}{(2\pi)^3 K M} \bar{\tilde{u}}(0; \mathbf{p}, \sigma) J^{\mu\nu} \tilde{u}(0; \mathbf{p}, \sigma) + \right. \\ & \left. + x^\mu j^{(21/12)\nu}(0; \mathbf{p}, \sigma) - x^\nu j^{(21/12)\mu}(0; \mathbf{p}, \sigma) \right] \quad , \end{aligned} \quad (95)$$

from which it follows that

$$\frac{\partial F^{\mu\nu}(x; \mathbf{p}, \sigma)}{\partial x^\nu} = q j^{(21/12)\mu}(x; \mathbf{p}, \sigma) \quad . \quad (96)$$

One can show that  $F^{\mu\nu}$  for a given momentum  $p^\mu$  transforms as a second rank tensor when  $p^\mu$  changes.

Equations (94) and (96) show that  $F^{\mu\nu}$  satisfies the Maxwell equations for a charged source current  $q j^{(21/12)\mu}(0; \mathbf{p}, \sigma) = q j^{(21/12)\mu}(x; \mathbf{p}, \sigma)$ , where the  $x$ -dependence cancels out in  $j^{(21/12)}$  as previously discussed. Thus  $F^{\mu\nu}$  is the electromagnetic field in the presence of the charge current  $q j^{(21/12)\mu}$ .

Other electromagnetic fields can have the same charge current density; they differ from  $F^{\mu\nu}$  by what are called ‘boundary conditions.’ Since the Maxwell equations are differential equations, any constant field, for example, can be added to  $F^{\mu\nu}$  and still obey the Maxwell equations for the same current source. In fact, the terms in parentheses with  $J^{\mu\nu}$  in (95) are constant. (In Appendix B, Problem 6, the constant field is matched to a dipole moment field, so these constant terms may be interpreted as being due to an intrinsic magnetic moment.)

It is important to note that the current is quadratic in coefficient function factors giving rise to interference terms when coefficient functions are summed. Furthermore, the specific properties of the canonical vectors  $a(\mathbf{p}, \sigma)$  may be relevant when coefficient functions for different momenta are mixed. Pursuing such considerations lie beyond the scope of this paper and may be treated elsewhere.

The coefficient function  $u^{21/12}$  may therefore be considered ‘intrinsically charged,’ meaning the charge arises from the Invariant Coefficient Hypothesis and the transformation properties of the spacetime symmetry group connected to the identity. Thus the Invariant Coefficient Hypothesis applied to covariant and unitary fields constrains  $u^{21/12}(x; \mathbf{p}, \sigma)$  to obey

the Dirac equation without assuming that the Dirac Equation applies and also determines the vector potential  $a^\mu$  and electromagnetic field  $F^{\mu\nu}$  that satisfy the Maxwell equations for the current of  $u^{21/12}(x; \mathbf{p}, \sigma)$  without assuming that the Maxwell equations hold.

## A Adjoint Representation

The canonical fields  $a(\mathbf{p}, \sigma)$  transform by a unitary representation of the Poincaré group, see equation (42) in the text. Inverses can also represent the group and the inverse of the unitary matrix  $D(W)$  representing the transformation  $W$  is the adjoint matrix,

$$D^{-1}(W) = D^\dagger(W) \quad . \quad (97)$$

Thus the construction of covariant field vectors  $\psi_l^{(12)}(x)$  and  $\psi_l^{(21)}(x)$  could equally well proceed with vectors that transform by the adjoint representation. Creation operators transform in this way, so in this Appendix, covariant fields are constructed from canonical field vectors that transform like creation operators.

Following standard notation, the canonical field vectors are denoted  $a^{c\dagger}(\mathbf{p}, \sigma)$ . The superscript  $c$  is a reminder that these may be a completely new set of vectors entirely distinct from the canonical field vectors  $a(\mathbf{p}, \sigma)$  used in the construction in the text. A covariant field vector  $\psi_l(x)$  constructed from the  $a^{c\dagger}(\mathbf{p}, \sigma)$  can be termed a ‘negative energy field’ while those of the text are ‘positive energy fields’.

The construction and derivation go through the same steps as the text, so only a few equations will be presented in this Appendix.

*Invariant Coefficient Hypothesis.* The covariant vector fields  $\psi^{(12)}$  and  $\psi^{(21)}$ , are required to be linear combinations of canonical field vectors  $a^{c\dagger}$ ,

$$\psi_l^{(12)}(x) = \sum_\sigma \int d^3p \, v_l^{(12)}(x; \mathbf{p}, \sigma) a^{c\dagger}(\mathbf{p}, \sigma) \quad , \quad (98)$$

and

$$\psi_l^{(21)}(x) = \sum_\sigma \int d^3p \, v_l^{(21)}(x; \mathbf{p}, \sigma) a^{c\dagger}(\mathbf{p}, \sigma) \quad . \quad (99)$$

The coefficient functions are now labeled  $v$ .

The covariant fields again transform by (40) and (41). The canonical field vectors  $a^{c\dagger}$  transform like creation operators, [13]

$$U(\Lambda, b) a^{c\dagger}(\mathbf{p}, \sigma) U^{-1}(\Lambda, b) = e^{i\Lambda p \cdot b} \sqrt{\frac{(\Lambda p)^t}{p^t}} \sum_{\bar{\sigma}} D_{\sigma\bar{\sigma}}^{(j)*}(W^{-1}(\Lambda, p)) a^{c\dagger}(\mathbf{p}_\Lambda, \bar{\sigma}) \quad , \quad (100)$$

where the symbols were defined with equation (42).

The dependence of  $v_l^{(12)}(x; \mathbf{p}, \sigma)$  on coordinates  $x$  and translation  $b$  can be taken care of by defining  $v_{\bar{l}}(\mathbf{p}, \sigma)$  in

$$v_l^{(12)}(x; \mathbf{p}, \sigma) = (2\pi)^{-3/2} e^{ip \cdot x} \sum_{\bar{l}} D_{\bar{l}l}^{(12)}(1, x) v_{\bar{l}}(\mathbf{p}, \sigma) \quad (101)$$

and

$$v_l^{(21)}(x; \mathbf{p}, \sigma) = (2\pi)^{-3/2} e^{ip \cdot x} \sum_{\bar{l}} D_{\bar{l}l}^{(21)}(1, x) v_{\bar{l}}(\mathbf{p}, \sigma) \quad . \quad (102)$$

The function  $v_{\bar{l}}(\mathbf{p}, \sigma)$  may be different for the 12- and 21- representations, but it satisfies the same equation in both representations,

$$\sum_{\bar{l}} D_{\bar{l}l}(\Lambda) v_{\bar{l}}(\mathbf{p}, \sigma) = \sqrt{\frac{(\Lambda p)^t}{p^t}} \sum_{\bar{\sigma}} v_l(\mathbf{p}_\Lambda, \bar{\sigma}) D_{\bar{\sigma}\sigma}^{(j)*}(W(\Lambda, p)) \quad . \quad (103)$$

Just as in the text the solution depends on two arbitrary constants which in general differ for the 12- and 21-representations. One finds that the coefficient functions  $v_l^{(12)}(x; \mathbf{p}, \sigma)$  and  $v_l^{(21)}(x; \mathbf{p}, \sigma)$  are given by

$$v_l^{(12)}(x; \mathbf{p}, \sigma) = D_{\bar{l}l}^{(12)}(1, x) v_{\bar{l}}(x; \mathbf{p}, \sigma) \quad (104)$$

$$v_l^{(21)}(x; \mathbf{p}, \sigma) = D_{\bar{l}l}^{(21)}(1, x) v_{\bar{l}}(x; \mathbf{p}, \sigma) \quad , \quad (105)$$

where  $v(x; \mathbf{p}, \sigma)$  is given by

$$v_{\bar{l}}(x; \mathbf{p}, \sigma) = (2\pi)^{-3/2} \sqrt{\frac{M}{p^t}} e^{-ip \cdot x} \sum_n D_{\bar{l}n}(L(p)) v_n(0, \sigma) \quad , \quad (106)$$

for both the 12- and 21-representations. The generators of the adjoint representation are determined within a similarity transformation and can be taken to be

$$-J^{(j)k*} = \frac{1}{2} \sigma^y \sigma^k \sigma^y \quad , \quad (107)$$

which is consistent with (50). The quantities  $v_n(0, \sigma)$  are found to be

$$v_l(0, +1/2) = \begin{pmatrix} 0 \\ d_+ \\ 0 \\ d_- \end{pmatrix} \quad (108)$$

and

$$v_l(0, -1/2) = - \begin{pmatrix} d_+ \\ 0 \\ d_- \\ 0 \end{pmatrix} , \quad (109)$$

where  $d_{\pm}$  are two arbitrary constants. Thus the coefficient functions  $v_l^{(12)}(x; \mathbf{p}, \sigma)$  and  $v_l^{(21)}(x; \mathbf{p}, \sigma)$  are determined by two parameters  $d_{\pm}$ . Of course, the constants  $d_{\pm}$  in  $v_n(0, \sigma)$  may be different for the 12- and 21-representations.

*Relating 12- and 21-Fields.* The 12/21 transition discussed in Section 3 can be extended to 12- and 21-fields just as in Section 5. One finds that

$$\gamma^4 v^{(12)}(x, \mathbf{p}, \sigma) = \tilde{v}^{(21)}(\tilde{x}, \tilde{\mathbf{p}}, \sigma) , \quad (110)$$

where

$$\tilde{v}^{(21)}(\tilde{x}, \tilde{\mathbf{p}}, \sigma) = (2\pi)^{-3/2} \sqrt{\frac{M}{\tilde{p}^t}} e^{i\tilde{p} \cdot \tilde{x}} D^{(21)}(L(\tilde{p}), \tilde{x}) \tilde{v}(0, \sigma) . \quad (111)$$

In these equations,

$$\tilde{p}^{\mu} = \eta_{\mu\nu} p^{\nu} = p_{\mu} \quad \text{and} \quad \tilde{x}^{\mu} = x_{\mu} \quad (112)$$

and

$$\tilde{v}(0, \sigma) = \gamma^4 v(0, \sigma) . \quad (113)$$

For  $\sigma = +1/2$  and the  $\gamma$ s in (1), equation (113) becomes

$$\tilde{v}(0, 1/2) = \begin{pmatrix} 0 \\ \tilde{d}_+ \\ 0 \\ \tilde{d}_- \end{pmatrix} = \gamma^4 v(0, 1/2) = \begin{pmatrix} 0 \\ d_- \\ 0 \\ d_+ \end{pmatrix} , \quad (114)$$

which implies that

$$\tilde{d}_{\pm} = d_{\mp} . \quad (115)$$

And it follows that

$$\tilde{v}_l(x; \mathbf{p}, \sigma, \tilde{d}_+, \tilde{d}_-) = v_l(x; \mathbf{p}, \sigma, d_-, d_+) , \quad (116)$$

much as found in Section 5.

*Translation Invariance; The Dirac Equation.* As in Section 6, define the ‘mixed’ quantity  $\psi^{(21/12)}$ ,

$$\psi^{(21/12)} = \frac{\alpha}{2} (1 - \gamma^5) \psi^{(12)} + \frac{\beta}{2} (1 + \gamma^5) \tilde{\psi}^{(21)} , \quad (117)$$

where  $\alpha$  and  $\beta$  are arbitrary complex constants. It follows that

$$U(\Lambda, b)\psi_l^{(21/12)}(x)U^{-1}(\Lambda, b) = D_{\bar{l}}^{-1}(\Lambda)\psi_l^{(21/12)}(\Lambda x + b) \quad , \quad (118)$$

and again one sees that  $\psi_l^{(21/12)}(x)$  is not invariant under the differential representation of translations since  $x \rightarrow \Lambda x + b$  depends on  $b$  but  $\psi_l^{(21/12)}(x)$  is invariant under matrix translations because  $D^{-1}(\Lambda)$  does not depend on  $b$ .

The choice of constants  $\alpha$  and  $\beta$  must be coordinated with the choice made in the text. The 12- and 21-fields still weigh equally but with opposite signs,

$$\alpha = -\beta = +1 \quad . \quad (119)$$

This time applying the 12/21 transition to  $\psi_l^{(21/12)}(x)$  brings back its negative. The choice produces a Dirac equation that has exactly the same form as the equation (78) determined in the text, including the sign.

The minus sign in (119) becomes the eigenvalue for the eigenvector  $v_l^{(21/12)}(0, \sigma)$  in

$$\gamma^4 v_l^{(21/12)}(0, \sigma) = -v_l^{(21/12)}(0, \sigma) \quad , \quad (120)$$

where  $v_l^{(21/12)}(0, \sigma)$  and similar quantities are defined in the same way that  $u_l^{(21/12)}(0, \sigma)$  and the quantities in the text were defined. This expression and the expression here similar to (70) in the text form the basis of the Dirac equation for free spin 1/2 particles with negative energy.

Following the same steps as those taken in the text, one finds that

$$\gamma^\mu p_\mu v_l^{(21/12)}(x; \mathbf{p}, \sigma) = -M v_l^{(21/12)}(x; \mathbf{p}, \sigma) \quad . \quad (121)$$

Note the minus sign in (121) for negative energy fields that does not appear in (77) for positive energy fields. The minus sign originates in the choice of constants  $\alpha$  and  $\beta$  in (119) and is needed because of the phase factor  $\exp(+ip \cdot x)$  that appears with the adjoint representation here compared with the phase factor  $\exp(-ip \cdot x)$  that appears with the representation in the text. Thus the momentum is  $p_\mu = -i\partial_\mu$  here which is the negative of the operator in the text for the positive energy fields. The minus sign in (119) adjusts for this difference and one finds that the negative energy field constructed from the adjoint representation in this way satisfies the same Dirac equation as the field in the text,

$$i\gamma^\mu \partial_\mu \psi_l^{(21/12)}(x) = M \psi_l^{(21/12)}(x) \quad . \quad (122)$$

The fields  $\psi_l^{(21/12)}(x)$  here and those in the text both transform as covariant field vectors and may be combined, perhaps with as-yet-arbitrary numerical coefficients, to make a covariant



field vector that is more general than either the negative energy field from Appendix A or the positive energy field from the text.

*Current as Vector Potential; the Maxwell Equations.* The calculation showing that the current obeys the Maxwell equations with  $j_i^{(21/12)}(\mathbf{p}, \sigma)$  as the source goes through just as in the text. The reason is that the result follows from the coordinate dependence of the following expressions

$$D^{(12)\dagger}(1, x) \gamma^4 \gamma^\mu D^{(12)}(1, x) \quad (123)$$

and

$$D^{(21)\dagger}(1, x) \gamma^4 \gamma^\mu D^{(21)}(1, x) \quad (124)$$

and these same expressions appear with the currents defined below.

Define the currents  $j^{(12)\mu}$  and  $\tilde{j}^{(21)\mu}$  by

$$j^{(12)\mu}(x; \mathbf{p}, \sigma) = \frac{p^t}{M} \bar{v}^{(12)}(x; \mathbf{p}, \sigma) \gamma^\mu v^{(12)}(x; \mathbf{p}, \sigma) \quad (125)$$

and

$$\tilde{j}^{(21)\mu}(x; \mathbf{p}, \sigma) = \frac{p^t}{M} \bar{\tilde{v}}^{(21)}(x; \mathbf{p}, \sigma) \gamma^\mu \tilde{v}^{(21)}(x; \mathbf{p}, \sigma) \quad , \quad (126)$$

where  $\bar{v} = v^\dagger \gamma^4$ . The current for  $q^\mu$  can be obtained from the current for  $p^\mu$  by applying  $L(q)L^{-1}(p)$  to the  $j^\mu$ s and the inverse of that transformation to  $x^\mu$ , as in Section 7.

As in the text, define the quantity  $a^\mu$  to be proportional to the sum of the currents,

$$a^\mu(x; \mathbf{p}, \sigma) = \frac{-q}{12K^2} [j^{(12)\mu}(x; \mathbf{p}, \sigma) + \tilde{j}^{(21)\mu}(x; \mathbf{p}, \sigma)] \quad , \quad (127)$$

where the constant  $q$  is introduced to put the following equations in a familiar form. Again it follows that  $a^\mu$  is a vector potential for the current  $qj^{(21/12)\mu}$  because one can show that it satisfies the equation

$$\partial^\tau \partial_\tau a^\mu(x; \mathbf{p}, \sigma) - \partial^\mu \partial_\kappa a^\kappa(x; \mathbf{p}, \sigma) = qj^{(21/12)\mu}(\mathbf{p}, \sigma) \quad , \quad (128)$$

where

$$j^{(21/12)\mu}(\mathbf{p}, \sigma) = \frac{p^t}{M} \bar{v}_i^{(21/12)}(x; \mathbf{p}, \sigma) \gamma^\mu v_i^{(21/12)}(x; \mathbf{p}, \sigma) \quad . \quad (129)$$

Equation (128) shows that  $a^\mu$  is the vector potential because  $a^\mu$  satisfies the Maxwell equation and that defines the vector potentials of the current  $j^{(21/12)\mu}$ .

The antisymmetric quantity  $F^{\mu\nu}$  is defined for the negative energy fields in this Appendix by

$$F^{\mu\nu} = a^{\mu,\nu} - a^{\nu,\mu} \quad , \quad (130)$$

and can be shown to satisfy the Maxwell equations (92),

$$F^{\mu\nu,\lambda} + F^{\nu\lambda,\mu} + F^{\lambda\mu,\nu} = 0 \quad (131)$$

and

$$\frac{\partial F^{\mu\nu}(x; \mathbf{p}, \sigma)}{\partial \nu} = qj^{(21/12)\mu}(\mathbf{p}, \sigma) \quad . \quad (132)$$

Equations (131) and (132) show that  $F^{\mu\nu}$  satisfies the Maxwell equations for a charged source current  $qj^{(21/12)\mu}(\mathbf{p}, \sigma)$ . Thus  $F^{\mu\nu}$  is the electromagnetic field in the presence of the charge current  $qj^{(21/12)\mu}$ . The coefficient functions  $u^{21/12}$  and  $v^{21/12}$  may therefore both be considered ‘intrinsically charged,’ meaning the charge arises from the Invariant Coefficient Hypothesis and the transformation properties of the spacetime symmetry group connected to the identity.

## B Problems

1. (a) Use the  $\gamma$ s in (1) and definitions (6), (8) and (10) to verify the matrix expressions (7), (9) and (11) for the Poincaré generators. (b) Show that these matrices satisfy the commutation rules (16), (17) and (18).
2. Show that  $-\sigma^{k*} = \sigma^y \sigma^k \sigma^y$ . Compare this with (50) and (107).
3. (a) Find the matrix  $D_{\bar{\sigma}\sigma}^{(j)}(W(\Lambda, p))$  to lowest order in  $\delta$  when  $p = M\{0, 0, \sinh \xi, \cosh \xi\}$  and  $\Lambda$  is a boost along  $x$  determined by  $\omega_{14} = \delta$ . Use the generators  $J^{(j)k} = \sigma^k/2$ , as in (50). (b) Also calculate the matrix  $D(\Lambda)$ , (19) and (21). (c) Use (a) and (b) to write equation (47) to first order in  $\delta$  for this case.
4. (a) Use the formulas in the text to find the electromagnetic field of a spin 1/2 particle at rest and with  $\sigma = +1/2$ . (b) Repeat (a) for a spin 1/2 particle moving in the  $z$ -direction with an energy of  $\cosh \xi$  times its rest energy,  $p^t = M \cosh \xi$ , and with  $\sigma = +1/2$ . (c) Show that the boost  $B(\xi \mathbf{z})$  transforms one field into the other.
5. In problem (4a), for a spin 1/2 particle at rest, the electric field was found to be

$$E^i = F^{4i} = \frac{2c_-^2}{3(2\pi)^3} q r^i \quad ,$$

where  $\mathbf{r} = \{x, y, z\}$ . Assume the covariant fields  $\psi_l^{(12)}(x)$  and  $\psi_l^{(21)}(x)$  are nonzero inside a sphere of radius  $R$  centered on the origin and vanish outside the sphere. Also assume that

the electric field external to the sphere obeys the Maxwell equations for empty space and is continuous at the boundary of the sphere. Then Gauss's Law holds with the surface integral of  $E^i$  over any surface containing the sphere being equal to the charge of the sphere divided by a constant,  $\oint E \cdot da = e/\epsilon_0$ , where  $e$  is the charge in coulombs and  $\epsilon_0$  is the permittivity constant. Use SI units. For convenience let  $q = e/4\pi\epsilon_0$  and show that

$$\frac{2c_-^2}{3(2\pi)^3} R^3 = 1 \quad ,$$

which normalizes  $\psi_l^{(21/12)}(x)$ , see (71).

6. In problem (4a), for a spin 1/2 particle at rest and with  $\sigma = +1/2$ , the magnetic field was found to be

$$B^z = F^{12} = \frac{2c_-}{3(2\pi)^3} \frac{c_+ q}{K} \quad .$$

Match this field to the magnetic dipole field due to a current confined to a ring of radius  $R$  in the  $xy$ -plane, by assuming the magnetic field is continuous at the edge of the ring,  $\sqrt{x^2 + y^2} = R$  and  $z = 0$ , where the field has the value found in problem (4a). This means that, at the edge of the ring, the magnetic field  $B^z$  has the value

$$B^z = \frac{\mu}{4\pi\epsilon_0 c^2} \frac{1}{R^3} \quad ,$$

where  $\mu$  is the magnetic moment of the ring and  $c$  is the speed of light ( $c = 1$  in the text). Assume that  $c_+ = c_-$  and show that, under these heuristic simplifying assumptions, the constant  $K$  is determined,

$$\frac{c}{K} = \frac{\hbar}{2mc} \quad ,$$

where the magnetic moment of a spin 1/2 Dirac particle of mass  $m$  and charge  $e$  is used,  $\mu = e\hbar/2m$ , with  $\hbar$  Planck's constant divided by  $2\pi$ . Thus, for this set of assumptions, the scale  $c/K$  for position dependent spin effects in the translation matrices  $D^{(12)}(1, x)$  and  $D^{(21)}(1, x)$ , (24) and (25), is one-half of a Compton wavelength.

7. (a) Let  $\alpha = \exp(-i\phi/2)/\sqrt{2}$  and  $\beta = \exp(+i\phi/2)/\sqrt{2}$  in the definition (65) of  $\psi^{(21/12)}(\alpha, \beta)$ , where the parameters are displayed explicitly. Find the matrix  $Dz(\phi)$  that transforms  $\psi^{(21/12)}(\alpha, \beta)$  to  $\psi^{(21/12)}(1/\sqrt{2}, 1/\sqrt{2})$  which is proportional to the  $\psi^{(21/12)}$  used in the text. (b) More generally, for  $\alpha = \sin(\theta/2)\exp(-i\phi/2)$  and  $\beta = \cos(\theta/2)\exp(+i\phi/2)$  find matrices  $\tau_x, \tau_y, \tau_z$ , with  $\tau_x\tau_y - \tau_y\tau_x = 2i\tau_z$  plus cyclic permutations, such that  $Dy(\theta) = \exp(-i\theta\tau_y/2)$  and  $Dz(\phi) = \exp(-i\phi\tau_z/2)$ , and

$$Dy(\pi/2)Dy(-\theta)Dz(\phi)\psi^{(21/12)}(\alpha, \beta) = \psi^{(21/12)}(1/\sqrt{2}, 1/\sqrt{2}) \quad ,$$

which is the same as the function used in the text except for a factor of the square root of two. (The commutation rules for the matrices  $\tau_k$  are the same as those for Pauli spin matrices, so the set of matrices  $\tau_k/2$ ,  $k \in \{x, y, z\}$ , are the generators of a representation of rotations in 3-dimensional Euclidean space with the  $Dk$  matrix representing rotations about the  $k$ -axis.)

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